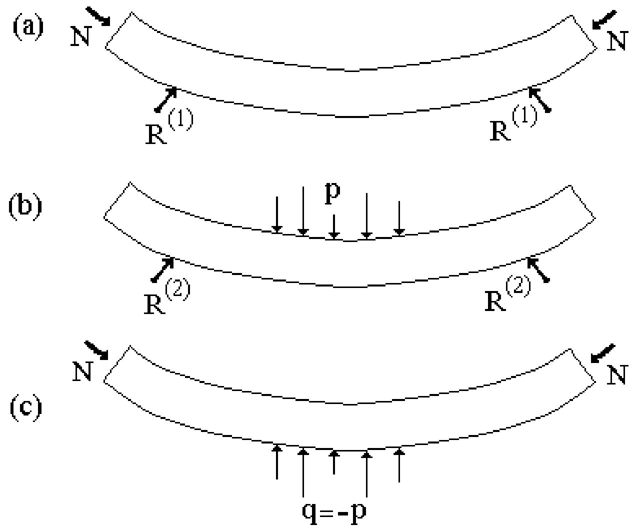

Fig.F4. (a) The equilibrium of a buckled plate acted by the forces $\pm\vec{N}$ and by the reactions of the supports.
(b) The equilibrium of a buckled plate acted by a vertical load p and by the reactions of the supports.

(c)The equilibrium of a buckled plate acted by the forces $\pm\vec{N}$ and by a load q having the same magnitude but a contrary sense with respect to the load presented in Fig.F4 (b).

In order to use the previous results, it is necessary to find when the mechanical state of stress / strain corresponding to the presence of the lateral forces is identical to the mechanical state of stress / strain corresponding to an unknown vertical load $p = p(x)$ (Fig.F4b). The equilibrium condition for the case shown in Fig.4a is

$$\sum \vec{N} + \sum \vec{R}^{(1)} = \vec{0} \quad , \tag{f65}$$

and the equilibrium condition for the case shown in Fig.F4b is

$$\sum \vec{p} + \sum \vec{R}^{(2)} = \vec{0} \quad , \tag{f66}$$

where $\sum \vec{R}^{(1)}$ and $\sum \vec{R}^{(2)}$ are the reactions of the supports in the above cases. Because the mechanical state of stress is identical in both cases, particularly in the neighbourhood of the supports, the reactions will be the same, i.e.

$$\sum \vec{R}^{(1)} = \sum \vec{R}^{(2)} . \tag{f67}$$

It follows that

$$\sum \vec{N} + \sum (-\vec{p}) = \vec{0} \quad . \tag{f68}$$

Hence for the corresponding state of stress / strain, the plate is in an equilibrium state if it is acted by the lateral forces and by a vertical load $\vec{q} = -\vec{p}$, in the absence of the supports (Fig.F4c). Consider a plate element having the horizontal length equal to dx and the ends denoted by A and B (Fig.F5).

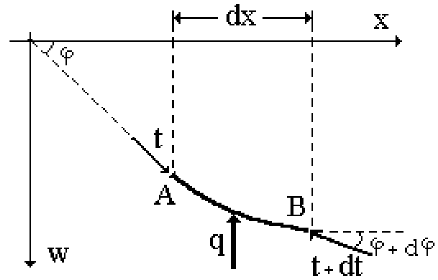

Fig.F5. The equilibrium of a plate element due to the load q and to the internal tensions.

Because the plate is a thin one, the tangential efforts are neglected. At the point A, it is acting a force (per unit length) denoted by $\vec{t}$, representing a normal effort, tangent to the plate. The angle between $\vec{t}$ and the horizontal axis is denoted by φ. At the point B, it is acting the effort $\overrightarrow{t+dt}$, making an angle equal to φ+dφ with the horizontal axis. The equilibrium conditions are

$$\begin{aligned} & t\cos\varphi-(t+dt)\cos(\varphi+d\varphi)=0 \\ & t\sin\varphi-(t+dt)\sin(\varphi+d\varphi)-q\,dx=0 \end{aligned} \qquad \text{(f69)}$$

For small angles φ, it follows that

$$\begin{aligned} & \cos\varphi\cong\cos(\varphi+d\varphi)\cong 1 \\ & \sin\varphi\cong\varphi\,,\ \sin(\varphi+d\varphi)\cong\varphi+d\varphi \\ & \varphi\cong tg\varphi=-\frac{dw}{dx} \end{aligned} \qquad \text{(f70)}$$

The first equation in (f69) gives

$$\frac{dt}{dx}=0 \qquad \text{(f71)}$$

Hence t has no variation along x-axis, i.e.

$$t=N \qquad \text{(f72)}$$

The second equation in (f69) gives

$$t\varphi-t(\varphi+d\varphi)-q\,dx=0 \qquad \text{(f73)}$$

By using (f74) and (f70c) it follows that

$$q=-N\frac{d\varphi}{dx}=N\frac{d^2w}{dx^2}\quad . \qquad \text{(f74)}$$

Hence the forces on the lateral faces of the plate are mechanical equivalent to a vertical load equal to

$$p=p(x)=-N\frac{d^2w}{dx^2}\quad . \qquad \text{(f75)}$$

It follows that eq. (f63) has the next general form

$$\frac{d^4w}{d\,x^4}+\frac{N}{D}\frac{d^2w}{d\,x^2}+\frac{4}{\alpha^4}w=\frac{P}{D} \qquad \text{(f76)}$$

F.11) The buckling of a simply leaning thin plate.

Consider the plate in Fig.F3. For simplicity, it is assumed that $c=0$ (i.e. the plate is leaning just at its ends). The buoyancy force and the vertical loads are neglected. Equation (f76) becomes

$$\frac{d^4w}{d\,x^4}+\frac{N}{D}\frac{d^2w}{d\,x^2}=0\quad , \qquad \text{(f77)}$$

together with the next conditions:

-at the end point having x=0: $w=0\,,\ d^2w\,/\,dx^2=0$ (f78)

-at the end point having x=a: $w=0\,,\ d^2w\,/\,dx^2=0$ (f79)

The equations (f77)-(f79) has the trivial solution $w\equiv 0$. It follows to find a *critical buckling value* $N=N^*$ in order the system (f77)-(f79) to have further non-trivial solutions. Successively, equation (f77) can be written as

$$\frac{d^2}{d\,x^2}\left(\frac{d^2w}{d\,x^2}+\frac{N}{D}w\right)=0\quad , \qquad \text{(f80)}$$

$$\frac{d^2w}{d\,x^2}+\frac{N}{D}w=C_1x+C_2\quad , \qquad \text{(f81)}$$

where C_1 , C_2 are two integration constants, vanishing according to (f78)-(f79). Hence (f81) is

$$\frac{d^2w}{dx^2}+\frac{N}{D}w=0 \quad , \tag{f82}$$

having the solution

$$w(x)=C_3\sin(\sqrt{N/D}x)+C_4\cos(\sqrt{N/D}x) \quad . \tag{f83}$$

From (f78) it follows that $C_4=0$, while from (f79) it follows the critical values

$$N_k^*=D(k\pi/a)^2 \ , \ k=1,2,\ldots \tag{f84}$$

The lowest critical value is obtained for $k=1$.
EXERCISES.
(1)Perform a study for the buckling of a 1-D plate having an embedded end point, the other being free.
(2)Perform a study for the buckling of a 1-D plate having both end points free. The plate is simply leaning at 1/3 from its length with respect to its left end.
(3)Perform a study of the simply leaning 1-D plate in the presence of the buoyancy force.
(4)Modify the equation of Sophie GERMAIN for the 2-D plate in the presence of lateral forces.
(5)Perform a study for the buckling a 2-D rectangular plate, simply leaning at all its sides.

F.12) The infinite extended 1-D plate.

By integrating both sides of eq.(f76) it follows

$$D\int_{-\infty}^{+\infty}\frac{d^4w}{dx^4}dx+N\int_{-\infty}^{+\infty}\frac{d^2w}{dx^2}dx+(\rho_m-\rho)g\int_{-\infty}^{+\infty}w(x)dx=\int_{-\infty}^{+\infty}P(x)dx \quad , \tag{f85}$$

Because w and its derivatives of any order are vanishing at infinite, the first two integrals in (f85) are vanishing too. It follows that the area bounded by the median curve (the flexure) and the horizontal x-axis is proportional to the load due to the relief, irrespective the presence of the lateral forces:

$$\int_{-\infty}^{+\infty}w(x)dx=\frac{1}{(\rho_m-\rho)g}\int_{-\infty}^{+\infty}P(x)dx \quad . \tag{f86}$$

Let an approximation of the relief be a set of m steps, each one of height equal to h_j and density equal to ρ_j, i.e.

$$P(x)=\sum_{j=1}^{m}P_j(x) \quad , \quad P_j(x)=\begin{cases}\rho_j g h_j, \text{ pentru } x\in\left[a_j,b_j\right]\\ 0\text{ , in rest}\end{cases} \quad . \tag{f87}$$

Equation (f86) becomes

$$\int_{-\infty}^{+\infty}w(x)dx=\frac{1}{\rho_m-\rho}\sum_{j=1}^{m}\rho_j h_j(b_j-a_j) \quad . \tag{f88}$$

Hence the area bounded by the flexural curve and the horizontal axis is a linear combination of the areas approximating the relief. In real cases, the flexural curve can be outlined along a finite interval denoted by $\left[-L\,,L\right]$, hence an upper bound for the difference of the densities can be obtained as

$$\rho_m-\rho<\sum_{j=1}^{m}\rho_j h_j(b_j-a_j)\Big/\int_{-L}^{+L}w(x)dx \quad . \tag{f89}$$

Equation (f76) will be solved by using FOURIER transforms.

F.13) FOURIER transforms. Properties.

The direct FOURIER transform (*t.F.d.*) of a function $f(x)$ is the new function Φ of variable u, defined as

$$\Phi[f](u)=\int_{-\infty}^{+\infty}f(x)\exp(-iux)dx\,, \quad i=\sqrt{-1} \tag{f90}$$

The inverse FOURIER transform (*t.F.i.*) of a function $\Phi(u)$ is the function $f(x)$ defined as

$$f(x) = \Phi^{-1}[\Phi[f](u)](x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi(u) \exp(+iux) du \tag{f91}$$

Differentiating both sides of eq. (f91) with respect to x, it follows that the *t.F.d.* of the first derivative $\frac{df}{dx}$ can be obtained by multiplying the the *t.F.d.* of $f(x)$ by iu . Hence the *t.F.d.* of the derivative $\frac{d^4 w}{dx^4}$ can be obtained by multiplying the *t.F.d.* of $w(x)$ by $(iu)^4 = u^4$. Consider two functions f and g of one variable. Their convolution product is

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(y) g(x-y) dy \quad . \tag{f92}$$

Permuting the integrals, it follows that the direct FOURIER transform of the convolution product is the product pf the transforms of both factors of the product, i.e.

$$\begin{aligned} \Phi[f * g](u) &= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(y) g(x-y) dy \right] \exp(-iux) dx \\ &= \int_{-\infty}^{+\infty} f(y) \exp(-iuy) dy \int_{-\infty}^{+\infty} g(z) \exp(-iuz) dz = \Phi[f] \; \Phi[g] \end{aligned} \quad . \tag{f93}$$

F.14) Solution of the flexure equation by using FOURIER transforms.

The solution of eq.(f76) is the sum of two terms, a term corresponding to the homogeneous equation and a term corresponding to a particular solution, i.e.

$$w(x) = w^h(x) + w^p(x) \tag{f94}$$

A particular solution $w^p = w^p(x)$ will be obtained applying the direct FOURIER transform to eq.(f76) and by using the above presented properties of the FOURIER transform

$$u^4 \Phi[w^p](u) - \frac{N}{D} u^2 \Phi[w^p](u) + \frac{4}{\alpha^4} \Phi[w^p](u) = \frac{1}{D} \Phi[P](u) \quad , \tag{f95}$$

i.e.

$$\Phi[w^p](u) = \frac{1}{D} \frac{\Phi[P](u)}{u^4 - Ku^2 + 4/\alpha^4} \quad . \tag{f96}$$

It is assumed that the value of the positive constant $K = N/D$ is small enough. Consider the particular case when the load due to the relief is a load concentrated at the origin of the axes, having the magnitude equal to unit. The direct FOURIER transform of this load is equal to unit too. The corresponding solution, denoted by w_G, represents the elastostatic GREEN function. It allows one to obtain the solution corresponding to an arbitrary load of magnitude equal to P . Hence

$$\Phi[w_G](u) = \frac{1}{D} \frac{1}{\left(u^2 - K/2\right)^2 + 4/\alpha^4 - (K/2)^2} \tag{f97}$$

It is assumed that the next condition is satisfied

$$|K| < 4/\alpha^2 \tag{f98}$$

Let

$$A = \sqrt{1/\alpha^2 + K/4} \quad , \quad B = \sqrt{1/\alpha^2 - K/4} \tag{f99}$$

But

$$\frac{1}{u^4 - Ku^2 + 4/\alpha^4} = \frac{\alpha^2}{8A}\left[\frac{u}{(u+A)^2+B^2} - \frac{u}{(u-A)^2+B^2}\right] + \frac{\alpha^2}{4}\left[\frac{1}{(u+A)^2+B^2} + \frac{1}{(u-A)^2+B^2}\right] \quad . \qquad \text{(f100)}$$

The next result is valid (Rîjic and Gradstein 1955)

$$\int_0^\infty \frac{z \operatorname{sgn} \operatorname{Re}(z)}{z^2 + x^2} \cos x dx = \frac{\pi}{2} \exp(-z) \quad , \quad z \neq 0 \quad . \qquad \text{(f101)}$$

By using (f101), the next inverse FOURIER transforms are obtained

$$\Phi^{-1}\left[\frac{1}{(u \pm A)^2 + B^2}\right](x) = \frac{1}{2B} \exp\left(-B|x| \mp iAx\right) \quad . \qquad \text{(f102)}$$

$$\Phi^{-1}\left[\frac{1}{(u+A)^2+B^2} + \frac{1}{(u-A)^2+B^2}\right](x) = \frac{1}{B} \exp\left(-B|x|\right) \cos\left(A|x|\right) \quad . \qquad \text{(f103)}$$

By using the property of the derivative, it follows

$$\Phi\left\{\frac{1}{2B}\frac{d}{dx}\left[\exp\left(-B|x| - iAx\right)\right]\right\}(u) = \frac{iu}{(u+A)^2+B^2} \quad . \qquad \text{(f104)}$$

In the same way

$$\Phi\left\{\frac{1}{2B}\frac{d}{dx}\left[\exp\left(-B|x| + iAx\right)\right]\right\}(u) = \frac{iu}{(u-A)^2+B^2} \quad . \qquad \text{(f105)}$$

Subtracting eq.(f104) from eq.(105), it follows that

$$\Phi\left\{\frac{1}{B}\frac{d}{dx}\left[\exp\left(-B|x|\right)\sin\left(Ax\right)\right]\right\}(u) = \frac{u}{(u-A)^2+B^2} - \frac{u}{(u+A)^2+B^2} \quad . \qquad \text{(f106)}$$

Hence

$$\Phi^{-1}\left[\frac{u}{(u-A)^2+B^2} - \frac{u}{(u+A)^2+B^2}\right] = \frac{1}{B}\frac{d}{dx}\left[\exp\left(-B|x|\right)\sin\left(Ax\right)\right] = \exp\left(-B|x|\right)\left[\frac{A}{B}\cos\left(A|x|\right) - \sin\left(A|x|\right)\right] \quad . \qquad \text{(f107)}$$

Using the above results, it follows after some elementary computations that

$$\Phi[w_G](u) = \frac{\alpha^2}{8D}\, \Phi\left\{\exp\left(-B|x|\right)\left[\frac{\sin\left(A|x|\right)}{A} + \frac{\cos\left(A|x|\right)}{B}\right]\right\} \quad , \qquad \text{(f108)}$$

$$w_G(x) = \frac{\alpha^2}{8D} \exp\left(-B|x|\right)\left[\frac{\sin\left(A|x|\right)}{A} + \frac{\cos\left(A|x|\right)}{B}\right] \quad , \qquad \text{(f109)}$$

It can be observed that $w_G \to \infty$ for $B \to 0$, corresponding to the buckling of the infinite plate in the presence of a lateral compressive stress. From (f96) and (f97) it follows that

$$\Phi[w^P] = \Phi[P]\Phi[w_G] \quad . \qquad \text{(f110)}$$

Hence the solution for an arbitrary load is the convolution of the load due to the relief and the function given by (f109), representing a general property of the GREEN function:

$$w^P(x) = (P * w_G)(x) = \int_{-\infty}^{+\infty} P(y)\, w_G(x-y) dy \ . \qquad \text{(f111)}$$

For the approximation of the relief represented by eq.(f87), it follows that

$$w^{p}(x)=\frac{1}{4(\rho_m-\rho)}\sum_{j=1}^{m}\rho_j h_j\left[I\left(b_j-x\right)-I\left(a_j-x\right)\right] \quad . \tag{f112}$$

where

$$I(z)=\mathrm{sgn}(z)\left\{\exp(-B|z|)\left[-2\cos(A|z|)+\frac{K}{\sqrt{(\rho_m-\rho)g/D-K^2/4}}\sin(A|z|)\right]+2\right\} \tag{f113}$$

The solution of the homogeneous equation can be immediately derived as

$$w^{h}(x)=[C_1\cos(Ax)+C_2\sin(Ax)]\exp(-Bx)+[C_3\cos(Ax)+C_4\sin(Ax)]\exp(Bx) \tag{f114}$$

Hence the general solution is

$$\begin{aligned} w(x)=\frac{1}{4(\rho_m-\rho)}\sum_{j=1}^{m}\rho_j h_j\left[I\left(b_j-x\right)-I\left(a_j-x\right)\right] \\ +[C_1\cos(Ax)+C_2\sin(Ax)]\exp(-Bx)+[C_3\cos(Ax)+C_4\sin(Ax)]\exp(Bx) \end{aligned} \tag{f115}$$

It follows to find the unknown coefficients C_1, C_2, C_3 and C_4 in some particular cases. For the infinite plate, the flexure W is subject to the next conditions:

$$\lim_{x\to\pm\infty} W(x)=0 \tag{f116}$$

Hence the coefficients C_1, C_2, C_3 and C_4 are vanishing and the general solution is just the particular solution represented by eq.(116). In the case of the semi-infinite plate the flexure W is subject, for example, to the next conditions:

$$\lim_{x\to\infty} W(x)=0 \tag{f117}$$

$$W(0+0)=W_0 \tag{f118}$$

$$\frac{d^2W}{dx^2}(0+0)=W_0''=-\mathbf{M}_0/D \tag{f119}$$

where W_0 , W_0'' and $\mathbf{M}_0$ (positive when acting into a clockwise sense) are the values of the flexure, that of the second derivative of the flexure and that of the bending moment respectively at the left end of the plate where the origin of the x-axis is selected. It follows

$$\begin{aligned} C_1 &= W_0-\sum_{j=1}^{m}\rho_j g h_j\left[I\left(b_j\right)-I\left(a_j\right)\right] \quad , \\ C_2 &= \frac{1}{2AB}\left[\frac{1}{4ABD}\sum_{j=1}^{m}\rho_j g h_j\left[\exp\left(-Ba_j\right)\sin\left(Aa_j\right)-\exp\left(-Bb_j\right)\sin\left(Ab_j\right)\right]-\frac{K}{2}C_1-W_0''\right] \end{aligned} \tag{f120}$$

and

$$C_3=C_4=0 \quad . \tag{f121}$$

A finite plate of variable thickness can be approximated in real cases by a sum of n elements having constant thickness and homogeneous elastic properties. To obtain the values of the unknown coefficients C_1, C_2, C_3 and C_4 for each element, proper conditions have to be verified at the ends of each element. A finite element algorithm based on the continuity of the values of the flexure, of its first derivative, of the bending moment and of the share force has been derived by Ivan (1997).

EXERCISE. Derive the expression of $w(x)$ for a load due to a relief having the equation

$$P=P(x)=\rho g h(x)=\begin{cases}\rho g h_0\sin(2\pi x/\lambda)\text{, pentru } & x\in[-\lambda/2,+\lambda/2] \\ 0 & \text{, in rest}\end{cases} \quad , \tag{f122}$$

where h_0 is the amplitude of the relief and λ is its wave-length.